通訊系統 (II)

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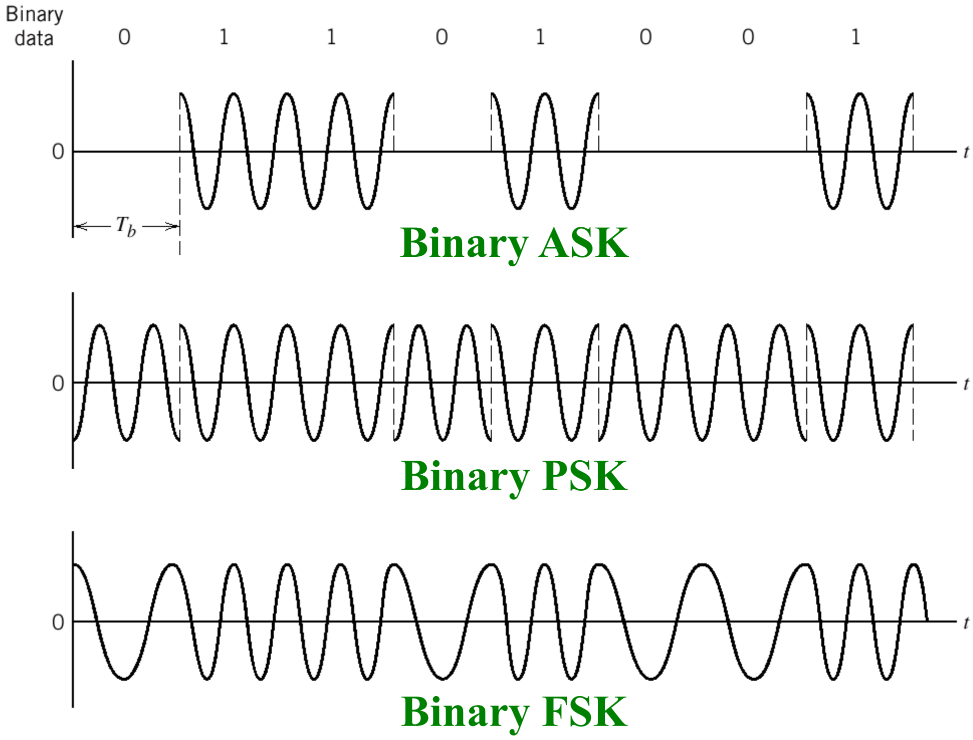
Chapter 2
Phase-Shift Keying Modulation

Introduction

Band-pass Digital Modulation

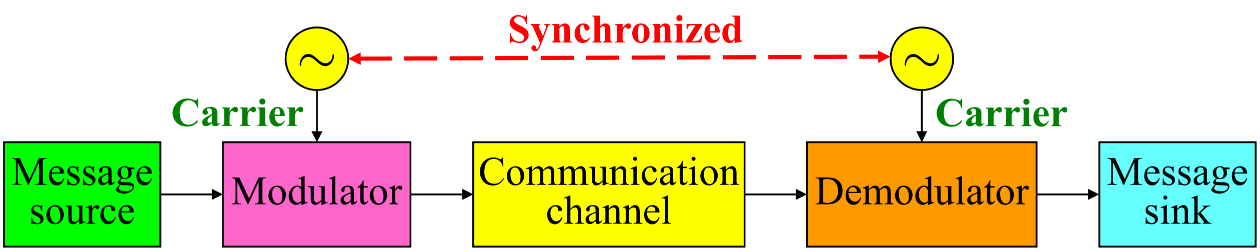
- In **digital band-pass transmission** (not baseband transmission), the incoming data stream is modulated onto a **carrier**
- The modulation process making the transmission possible involves switching (keying) the **amplitude**, **frequency**, or **phase** of a sinusoidal carrier
  - **Amplitude-shift keying (ASK)**
  - **Phase-shift keying (PSK)**
  - **Frequency-shift keying (FSK)**

Band-pass Digital Modulation



Coherent and Non-coherent

- Digital modulation techniques may be classified into **coherent** and **noncoherent** techniques
  - Depending on whether a **phase-recovery circuit** (or a **reference signal** for carrier) is required **at the receiver** (for data detection) or not
  - The circuit ensures that the **local carrier at the receiver** is **synchronized** (in both **frequency and phase**) to the carrier used to **modulate** the incoming data **at the transmitter**



M-ary Modulation

- In an ***M*-ary** modulation scheme, multiple bits are transmitted in a symbol
  - $n = \log_2 M$ bits/symbol
- The signal are generated by changing the amplitude, phase, or frequency of a sinusoidal carrier in ***M* discrete steps**
- The M-ary signals can also be generated by combining **different** modulation methods into a **hybrid form**
  - M-ary **amplitude-phase** keying (APK)
- A special form of M-ary APK is M-ary **quadrature-amplitude modulation (QAM)**

Basic Assumptions

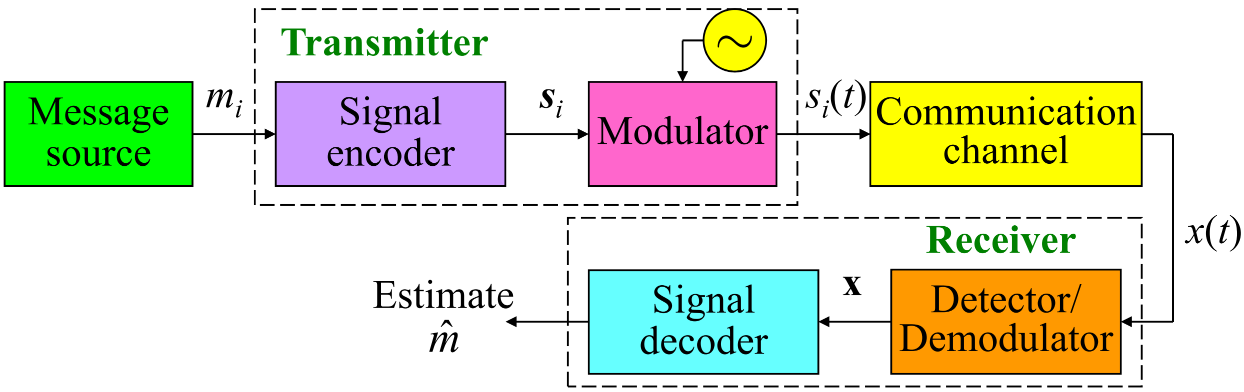
- For the original data sequence, we assume that
  - The transmitted **bit rate** is R_b, fixed for different modulation
  - The **bit duration** is fixed as $T_b = 1/R_b$
  - The **energy per bit** is set as E_b
- In a binary modulation scheme with $M = 2$
  - The transmitted **symbol rate** is $R = R_b$
  - The **symbol duration** is $T = T_b$
  - The **energy per symbol** is $E = E_b$
- In an ***M*-ary** modulation scheme with $n = \log_2 M$ bits/symbol
  - The transmitted **symbol rate** is $R = R_b/n$
  - The **symbol duration** is $T = n \times T_b$
  - The **energy per symbol** is $E = n \times E_b$

Band-pass Transmission Model

Band-pass Transmission Model

- A message source emits one symbol every T seconds, with the symbols belonging to an alphabet of M symbols: $m_1, m_2, \ldots, m_M$
- The ***a priori* probabilities** $p_1, p_2, \ldots, p_M$ specify the message source output
  - If the M symbols of the alphabet are **equally likely**

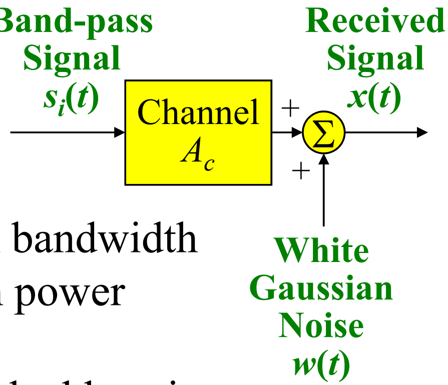
$$p_i = P(m_i) = 1/M \quad \text{for } i = 1, 2, \cdots, M$$



Band-pass Transmission Model (Cont.)

- The **signal encoder** produces a corresponding **signal vector** $\boldsymbol{s}_i$ made up of N (the signal space **dimension**) real elements
- The **modulator** constructs a distinct signal $s_i(t)$ of duration T
  - The signal $s_i(t)$ is a **real-valued energy signal**

$$E_i = \int_0^T s_i^2(t)\,dt$$



- The communication channel is assumed to have the two characteristics:
  - The channel is **linear**, with an enough bandwidth
  - The channel noise $w(t)$ is AWGN with power spectral density $N_0/2$
- The channel only attenuates the signal and adds noise
  - **No distortion** $\quad x(t) = \alpha s_i(t) + w(t), \quad 0 \le t \le T$

Band-pass Signal Representation

- A band-pass signal $g(t)$ can be represented as its equivalent **complex envelope** $\tilde{g}(t)$
  - The carrier component contains **no information** about $g(t)$

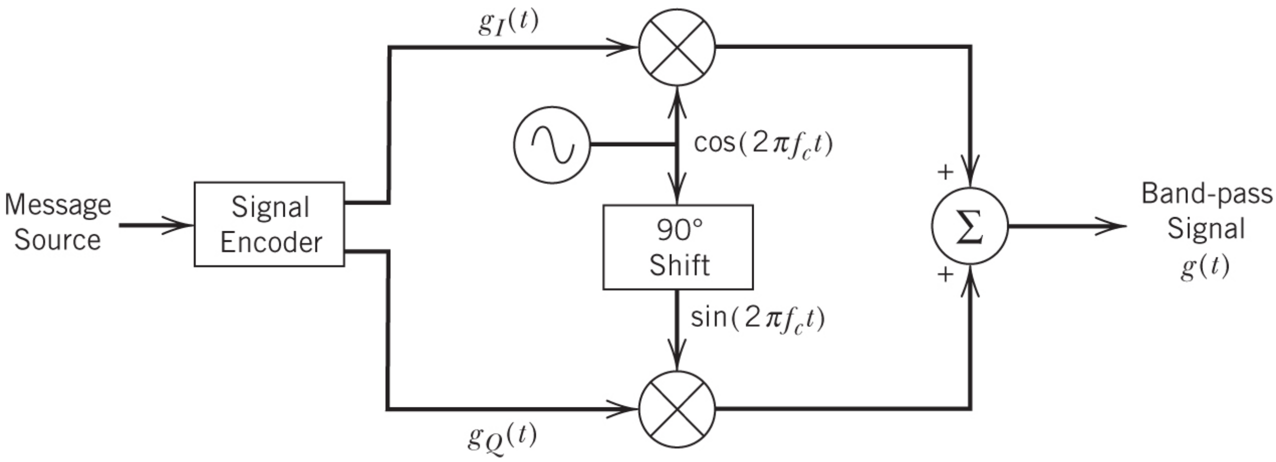
$$g(t) = \Re\left[\tilde{g}(t)e^{j2\pi f_c t}\right] = \Re\left[\left(g_I(t) + jg_Q(t)\right)e^{j2\pi f_c t}\right]$$

- Thus, we can represent $g(t)$ as its low-pass **in-phase** and **quadrature** components
  - In-phase component: $g_I(t)$
  - Quadrature component: $g_Q(t)$

  } **Orthogonal**

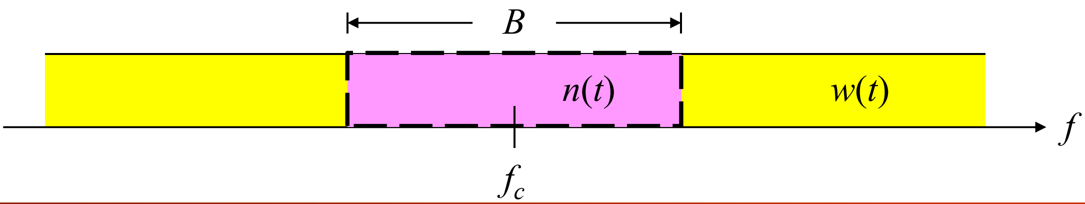
Band-pass Signal Representation – Tx

- At the transmitter, the band-pass signal $g(t)$ can be generated by using its **low-pass in-phase** and **quadrature** components
  - In-phase: $g_I(t)$ used to modulate the carrier $\cos(2\pi f_c t)$
  - Quadrature: $g_Q(t)$ used to modulate the carrier $\sin(2\pi f_c t)$



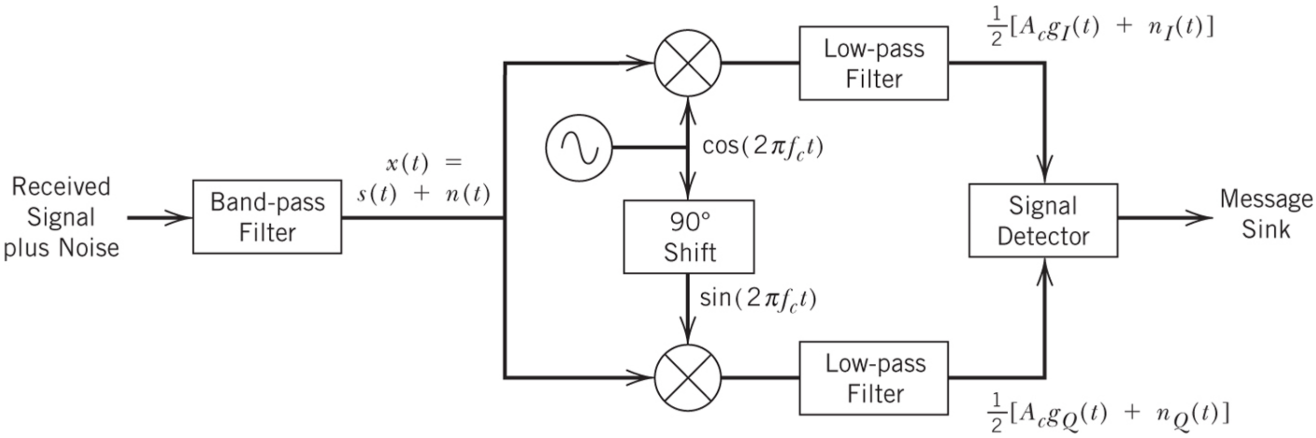
Band-pass Signal Representation – Rx

- A receiver includes a **band-pass filter** at the front end
  - The bandwidth must be **just large enough** to pass the transmitted signal, but not to admit excessive noise
  - That is, set the filter bandwidth to the **signal bandwidth** B
  - The white noise is converted to **narrowband noise** $n(t)$
- The narrowband noise $n(t)$ can also be represented as its equivalent **complex envelope** $\tilde{n}(t)$
  - In-phase component: $n_I(t)$
  - Quadrature component: $n_Q(t)$



Band-pass Signal Representation – Rx (Cont.)

- The signal is down-converted by the local orthogonal carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$
- After passing a **low-pass filter**, the signals used for detection are $\frac{1}{2}[A_c g_I(t)+n_I(t)]$; $\frac{1}{2}[A_c g_Q(t)+n_Q(t)]$

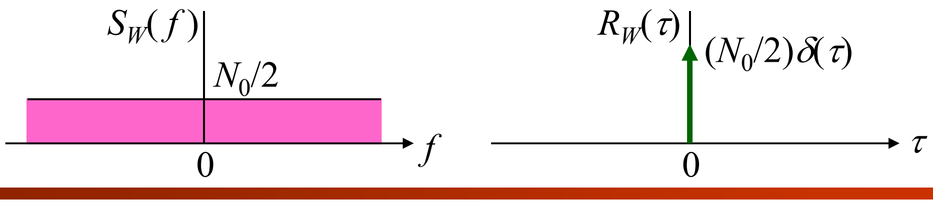


White Noise

- **White noise:**
  - An idealized form of noise
  - The power spectral density is **independent** of the operating frequency
- The power spectral density of white noise is

$$S_W(f) = \frac{N_0}{2} \quad \text{(watts/Hz)}$$

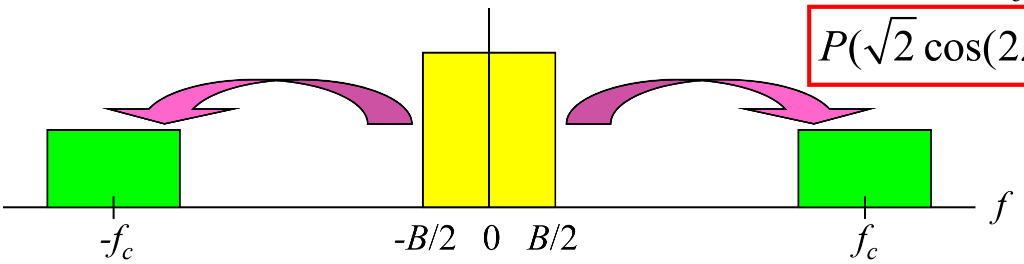
$$R_W(\tau) = \frac{N_0}{2}\delta(\tau)$$



Narrowband Noise

- The **narrowband** noise is equivalent to a **low-pass filtered** white noise multiplied by a sinusoidal wave $\sqrt{2}\cos(2\pi f_c t)$

$$P(\sqrt{2}\cos(2\pi f_c t)) = 1$$



- Considering the **narrowband noise** $n(t)$ of bandwidth B centered on f_c, it can be decomposed into two components
  - The two **orthogonal** bases: $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$
  - The in-phase component: $n_I(t)$ (**low-pass signal**)
  - The quadrature component: $n_Q(t)$ (**low-pass signal**)

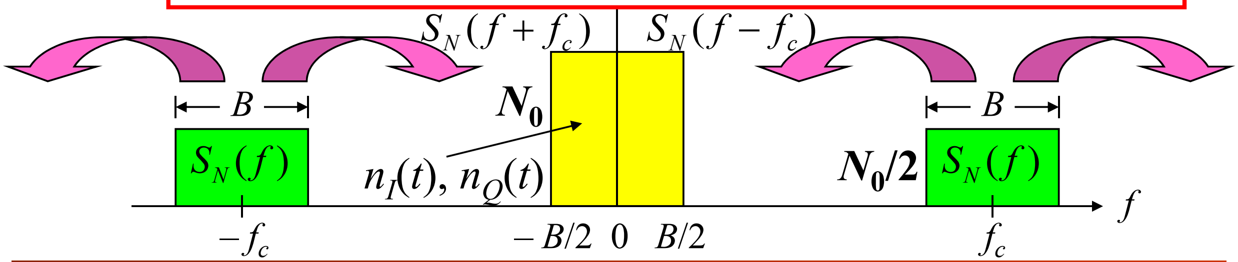
$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

Narrowband Noise (Cont.)

- Since $n(t)$ have zero mean, both $n_I(t)$ and $n_Q(t)$ have **zero mean**
- If $n(t)$ is Gaussian, $n_I(t)$ and $n_Q(t)$ are **jointly Gaussian**
  - The properties of Gaussian process
- If $n(t)$ is stationary, $n_I(t)$ and $n_Q(t)$ are **jointly stationary**
- $n_I(t)$ and $n_Q(t)$ have the same power spectral density

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B/2 \le f \le B/2 \\ 0, & \text{otherwise} \end{cases}$$

Low-pass $n_I(t) = n(t) \times 2\cos(2\pi f_c t); \quad n_Q(t) = n(t) \times (-2\sin(2\pi f_c t))$



Narrowband Noise (Cont.)

- Both $n_I(t)$ and $n_Q(t)$ have **the same variance** as $n(t)$

$$N_0 B = \left(N_0/2\right)\times 2B$$

- The cross-spectral density is purely **imaginary**

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & -B/2 \le f \le B/2 \\ 0, & \text{otherwise} \end{cases}$$

- If $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is **symmetric** about f_c, $n_I(t)$ and $n_Q(t)$ are **statistically independent**
  - Since $S_N(f)$ is symmetric about f_c, we have

$$S_{N_I N_Q}(f) = S_{N_Q N_I}(f) = j[S_N(f+f_c) - S_N(f-f_c)] = 0$$

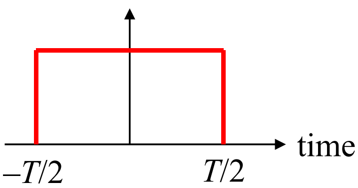
Pulse-Shaping Filter

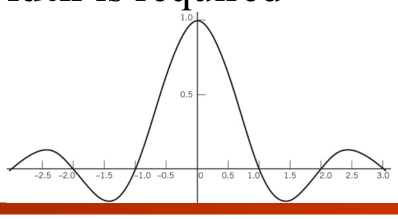
Pulse-Shaping

- In the **modulator**, the signal characteristics depend on the output signal waveform of $s_i(t)$
- **Pulse shaping** is the process of changing the **pulse waveform** of the output signal for transmission
  - Make the signal possessing the desired characteristics and suitable for the communication channels
  - For example: to reduce signal bandwidth, to eliminate inter-symbol interference (ISI), to enhance the transmission efficiency, etc.
- In general, there are two important requirements for the pulse-shaping filter used in wireless communications systems
  - **Bandwidth limitation** and **ISI elimination**

Pulse-Shaping Filter

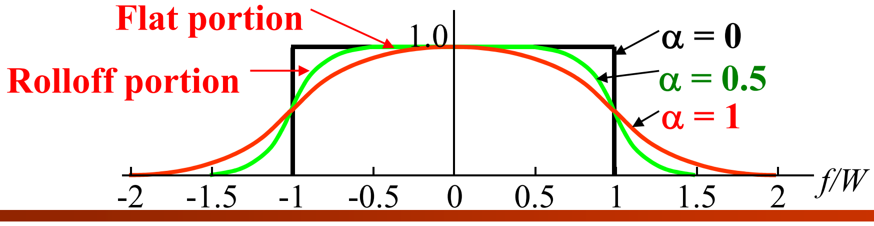
- There are different types of pulse-shaping filter:
  - Rectangular shaped filter
  - Sinc shaped filter
  - Raised-cosine filter
  - Gaussian filter
- Rectangular shaped filter: generate a signal with a rectangular pulse waveform during the symbol duration
  - **No ISI** is introduced in the transmitted signal
  - **Infinite channel bandwidth** is required





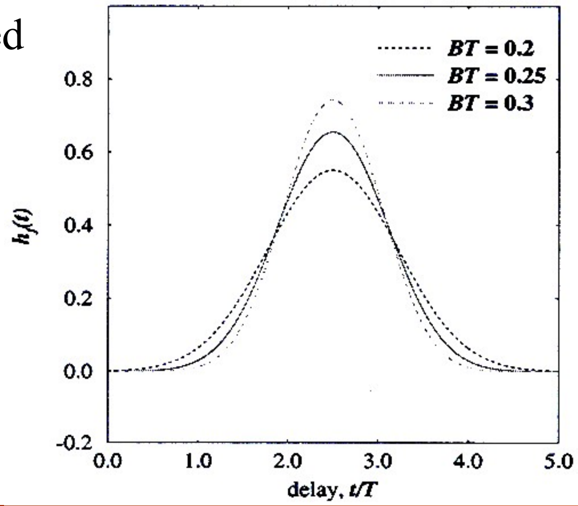
Pulse-Shaping Filter (Cont.)

- Sinc shaped filter: generate a signal with a strictly limited signal bandwidth
  - **Time overlapping (ISI)** is introduced
  - **Finite channel bandwidth** is required
- Raised-cosine filter: generate a signal with a raised cosine spectrum
  - Restrict the signal duration in the time domain
  - Restrict the signal bandwidth in the frequency domain
  - **ISI** is introduced



Pulse-Shaping Filter (Cont.)

- Gaussian filter: generate a signal with a Gaussian-like signal shape
  - Restrict the signal duration in the time domain
  - Restrict the signal bandwidth in the frequency domain
  - **ISI** is introduced



Coherent Phase-Shift Keying

Binary Phase-Shift Keying

Binary Phase-Shift Keying

- In coherent binary PSK, the pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols **1** and **0** is defined by

$$s_1(t) = \sqrt{2E_b/T_b}\cos(2\pi f_c t)$$
$$s_2(t) = \sqrt{2E_b/T_b}\cos(2\pi f_c t + \pi) = -\sqrt{2E_b/T_b}\cos(2\pi f_c t)$$

  – where $0 \le t < T_b$, and E_b is the **signal energy per bit**

  – The carrier frequency f_c is chosen equal to n_c/T_b for some **fixed integer** n_c (Each symbol contains an integral number of cycles of the carrier: **to maintain I-Q orthogonality**)

  – **Antipodal signals:** a pair of sinusoidal waves that differ only in a relative phase-shift of **180°**

- There is only **one basis function** of unit energy

$$\phi_1(t) = \sqrt{2/T_b}\cos(2\pi f_c t), \quad 0 \le t < T_b$$

Binary Phase-Shift Keying (Cont.)

- The pair of signals $s_1(t)$ and $s_2(t)$ can be expressed as follows:

$$s_1(t) = \sqrt{E_b}\,\phi_1(t) = s_{11}\phi_1(t), \quad 0 \le t < T_b$$
$$s_2(t) = -\sqrt{E_b}\,\phi_1(t) = s_{21}\phi_1(t), \quad 0 \le t < T_b$$

  – The **dimension** of the signal space is $N = 1$

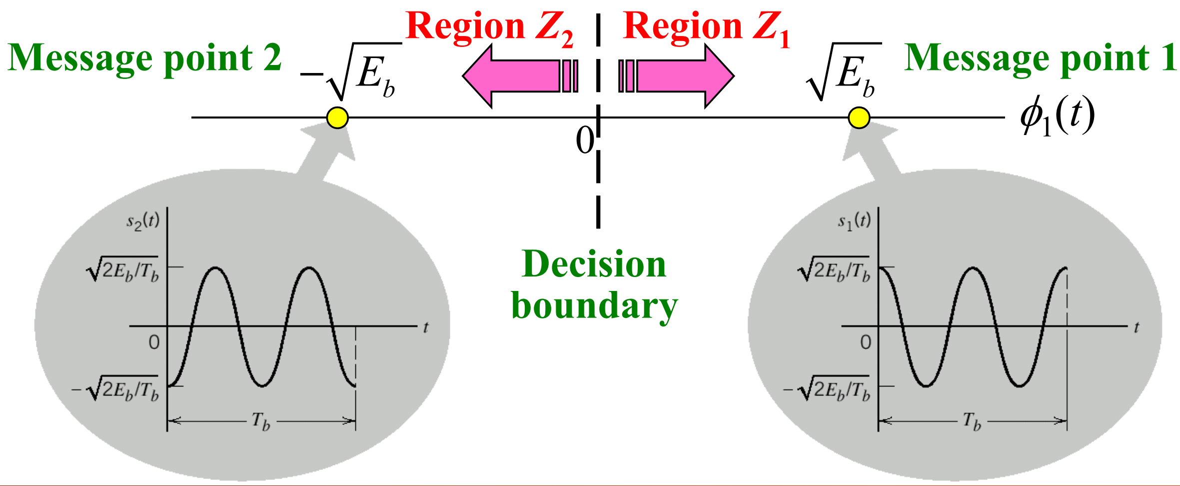
- The coordinates of the message points are

$$s_{11} = \int_0^{T_b} s_1(t)\phi_1(t)\,dt = +\sqrt{E_b}$$
$$s_{21} = \int_0^{T_b} s_2(t)\phi_1(t)\,dt = -\sqrt{E_b}$$

Error Probability of Binary PSK

- Based on the **ML decision rule**, the decision regions are

  $Z_1: 0 < x_1 < \infty; \; Z_2: -\infty < x_1 < 0; \quad \text{where} \quad x_1 = \int_0^{T_b} x(t)\phi_1(t)\, dt$
  - Z_1: The set of points closest to message point 1
  - Z_2: The set of points closest to message point 2



Error Probability of Binary PSK (Cont.)

- The **decision random variable** X_1 is the correlation of the received signal $x(t)$ and the **basis function** $\phi_1(t)$

$$X_1 = \int_0^{T_b} x(t)\phi_1(t)\, dt = \int_0^{T_b} \left[s_i(t) + n(t) \right] \phi_1(t)\, dt$$
$$= s_{i1} + \int_0^{T_b} n(t)\phi_1(t)\, dt$$
$$= s_{i1} + \sqrt{1/2T_b} \int_0^{T_b} n_I(t)\, dt$$

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$
$$\phi_1(t) = \sqrt{2/T_b}\cos(2\pi f_c t)$$

- Because $n_I(t)$ is a **zero-mean** Gaussian process with a PSD N_0 for $-B/2 \le f \le B/2$
  - X_1 is a **Gaussian distributed** random variable
    - Mean: $\mu = s_{i1}$
    - Variance: $\sigma^2 = 1/2T_b \times N_0 B \times T_b^2 = N_0/2$
      - The power of $n_I(t)$ is $N_0 B$ and $B = 1/T_b$

Error Probability of Binary PSK (Cont.)

- The conditional pdf of random variable X_1, given that **symbol 1** was transmitted, is defined by

$$f_{X_1}(x_1|1) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{11})^2\right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}\left(x_1 - \sqrt{E_b}\right)^2\right]$$

- The conditional pdf of random variable X_1, given that **symbol 0** was transmitted, is defined by

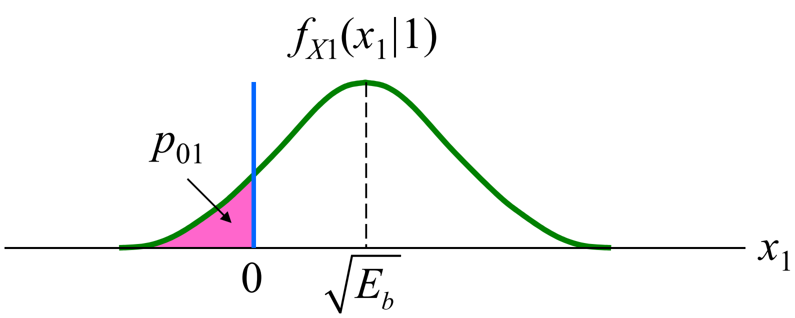
$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}\left(x_1 + \sqrt{E_b}\right)^2\right]$$

Error Probability of Binary PSK (Cont.)

- The conditional probability of the receiver deciding in favor of **symbol 0**, given that **symbol 1 was transmitted**, is

$$p_{01} = \int_{-\infty}^{0} f_{X_1}(x_1|1)\, dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} \exp\left[-\frac{1}{N_0}\left(x_1 - \sqrt{E_b}\right)^2\right] dx_1$$

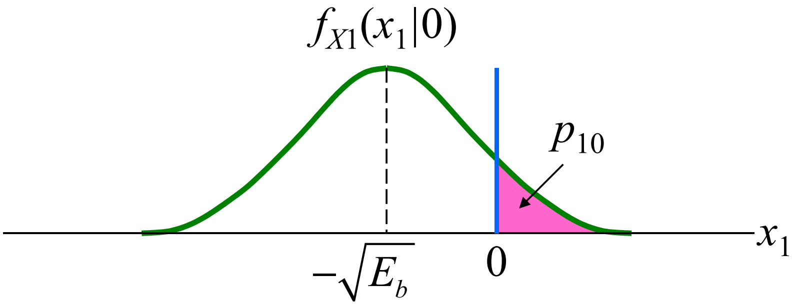


PDF of X_1 when '1' is transmitted

Error Probability of Binary PSK (Cont.)

- The conditional probability of the receiver deciding in favor of **symbol 1**, given that **symbol 0 was transmitted**, is

$$p_{10} = \int_0^\infty f_{X_1}\left(x_1 \middle| 0\right) dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[-\frac{1}{N_0}\left(x_1 + \sqrt{E_b}\right)^2\right] dx_1$$



PDF of X_1 when '0' is transmitted

Error Probability of Binary PSK (Cont.)

- Putting $z = \left(x_1 - \sqrt{E_b}\right) / \sqrt{N_0}$ and changing the variable of integration to z, we have

$$p_{01} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\sqrt{E_b/N_0}} \exp\left[-z^2\right] dz = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/N_0}\right)$$

  - where erfc(·) is the **complementary error function**
- Similarly, the conditional probability p_{10} of the receiver deciding in favor of symbol 1, given that symbol 0 was transmitted, has **the same value** as p_{01}
- Assuming $p_1 = p_0$, the average probability of symbol error or, equivalently, the bit error rate for coherent binary PSK is

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/N_0}\right)$$

Error Probability of Binary PSK (Cont.)

- **Complementary error function**:

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, dz$$

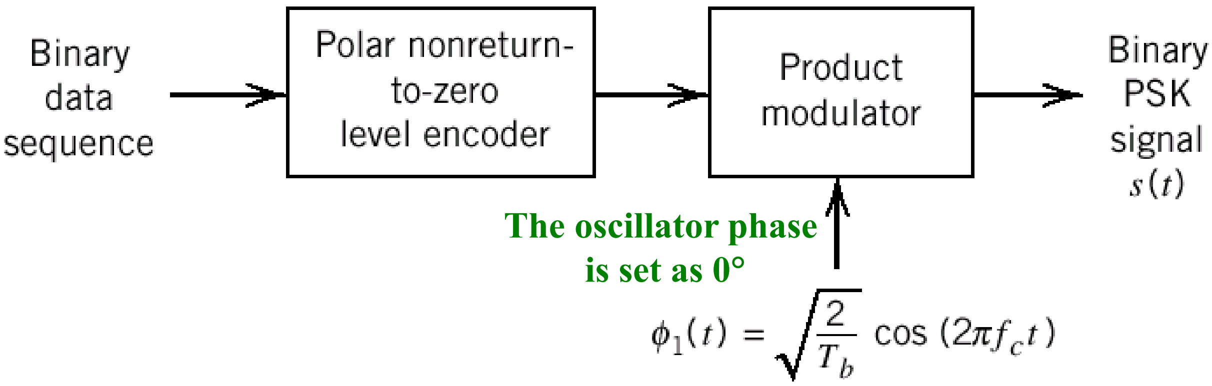
$$\boxed{Q(u) = \frac{1}{2}\mathrm{erfc}\left(\frac{u}{\sqrt{2}}\right)}$$

- For **large positive** values of u, we have an **upper bound**

$$\mathrm{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

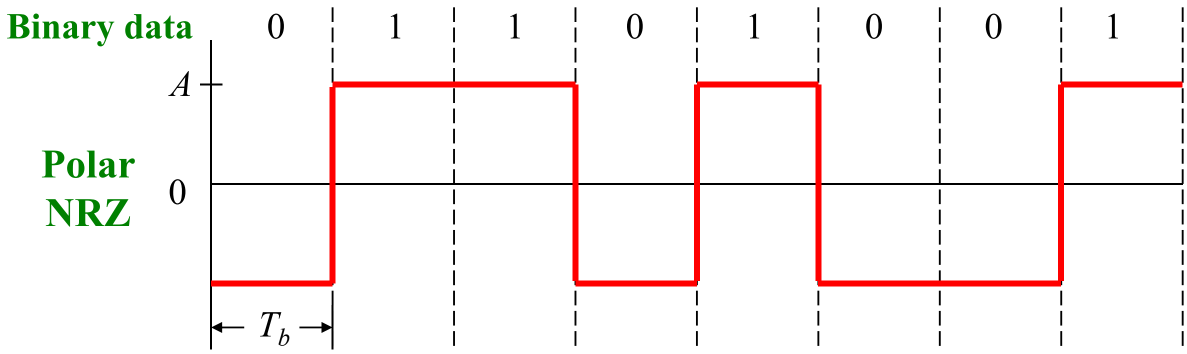
Generation of Binary PSK Signals

- The signal transmission encoding is performed by a **polar nonreturn-to-zero (NRZ)** level encoder
- The resulting binary wave and a sinusoidal carrier $\phi_1(t)$ are applied to a **product modulator** to produce the binary PSK signal



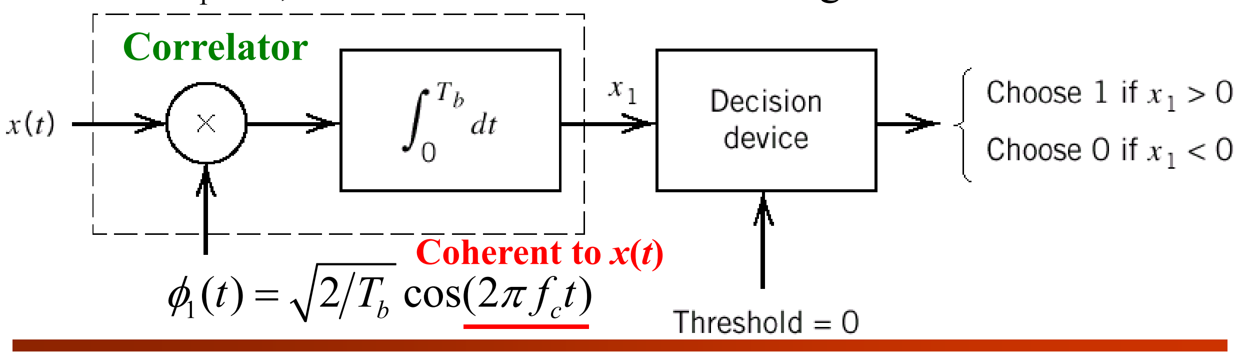
Polar NRZ

- **Polar nonreturn-to-zero (NRZ)** signaling:
  - Symbol 1: a pulse of amplitude ***A*** for the duration of the symbol
  - Symbol 0: a pulse of amplitude **–*A*** for the duration of the symbol



Detection of Binary PSK Signals

- The received noisy PSK signal $x(t)$ is applied to a **correlator**
  - which is supplied with a locally generated **coherent** reference signal $\phi_1(t)$ (**Coherent detection** is necessary)
- The output x_1 is compared with a **threshold** of **zero volts**
  - If $x_1 > 0$, the receiver decides in favor of **symbol 1** ($s_1(t)$)
  - If $x_1 < 0$, the receiver decides in favor of **symbol 0** ($s_2(t)$)
  - If $x_1 = 0$, the receiver makes a random guess



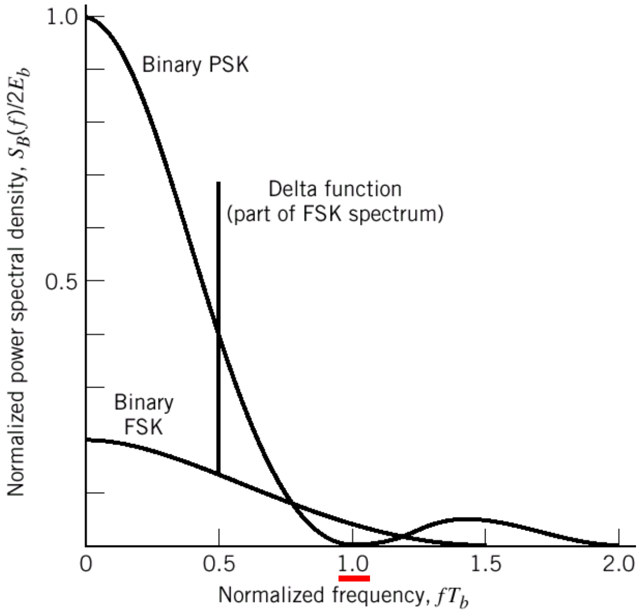
Power Spectra of Binary PSK Signals

- The complex envelope of a binary PSK signal only consists of an **in-phase component**
  - Symbol 1: + $g(t)$
  - Symbol 0: – $g(t)$
- The symbol **shaping function**:

$$g(t) = \begin{cases} \sqrt{2E_b/T_b}, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

- The energy spectral density is the **squared magnitude** of the signal's Fourier transform

$$S_B(f) = \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2} = 2E_b \,\text{sinc}^2(T_b f)$$



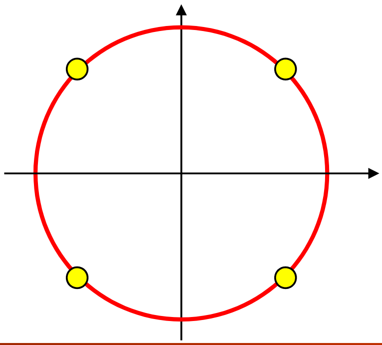
QuadriPhase-Shift Keying (QPSK)
Quadrature Phase-Shift Keying

QuadriPhase-Shift Keying (QPSK)

- Quadriphase-Shift Keying is used to improve **bandwidth efficiency** ⇒ an example of **quadrature-carrier multiplexing**
- The two **orthonormal basis functions** are

$$\phi_1(t) = \sqrt{2/T}\cos(2\pi f_c t); \quad \phi_2(t) = \sqrt{2/T}\sin(2\pi f_c t)$$

- In QPSK, the phase of the carrier takes on one of four **equally spaced** values, such as **π/4, 3π/4, 5π/4, and 7π/4**



Signal Space of QPSK

- The transmitted signal is defined as

$$s_i(t) = \begin{cases} \sqrt{2E/T}\cos(2\pi f_c t + (2i-1)\pi/4), & 0 \le t < T \\ 0, & \text{elsewhere} \end{cases}, \; i = 1, 2, 3, 4$$

  - $E = 2E_b$ is the symbol energy; $T = 2T_b$ is the symbol duration
- Based on $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\phi_1(t) \Rightarrow s_I$
$\phi_2(t) \Rightarrow -s_Q$

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left[(2i-1)\frac{\pi}{4}\right]\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin\left[(2i-1)\frac{\pi}{4}\right]\sin(2\pi f_c t)$$

⇒ The two **basis functions**

$$\phi_1(t) = \sqrt{2/T}\cos(2\pi f_c t)$$
$$\phi_2(t) = \sqrt{2/T}\sin(2\pi f_c t)$$

⇒ The signal vectors are

$$\mathbf{s}_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\pi/4\right) \\ -\sqrt{E}\sin\left((2i-1)\pi/4\right) \end{bmatrix}$$

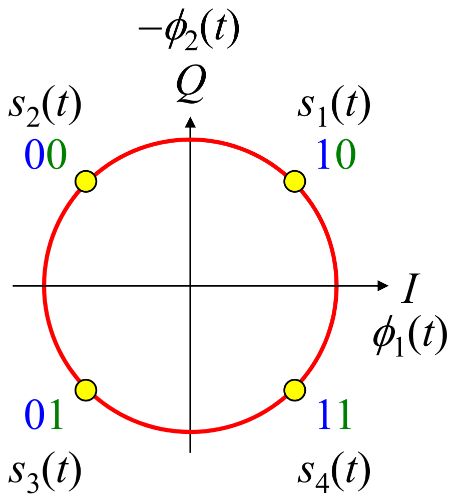
Signal Space of QPSK (Cont.)

- **Gray-encoding** is used
  - Only **one bit** is changed from one **dibit** to the next

Gray-encoded input dibit	Phase of QPSK	Message Points s_{i1}	s_{i2}
10	π/4	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	3π/4	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	5π/4	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	7π/4	$+\sqrt{E/2}$	$+\sqrt{E/2}$

I-channel Q-channel

$$\mathbf{s}_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right) \\ -\sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right) \end{bmatrix}$$

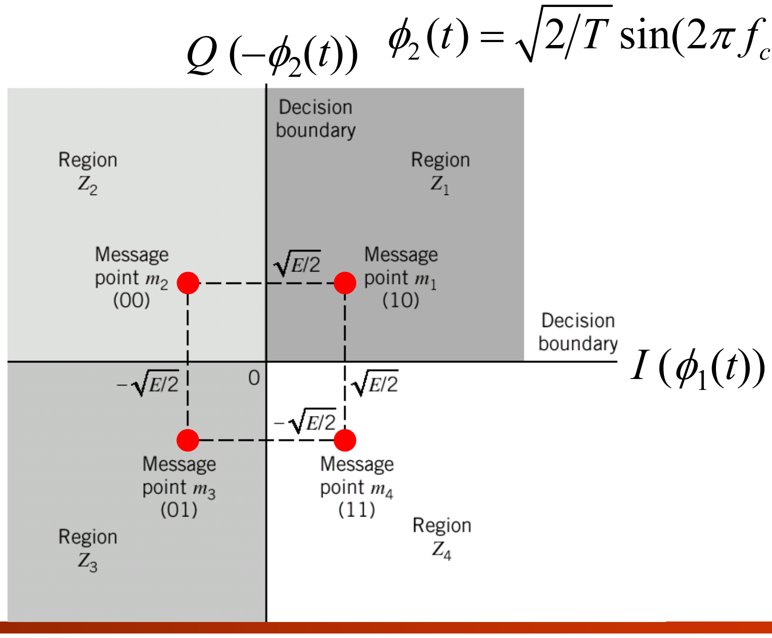


IQ-plane

Signal Space of QPSK (Cont.)

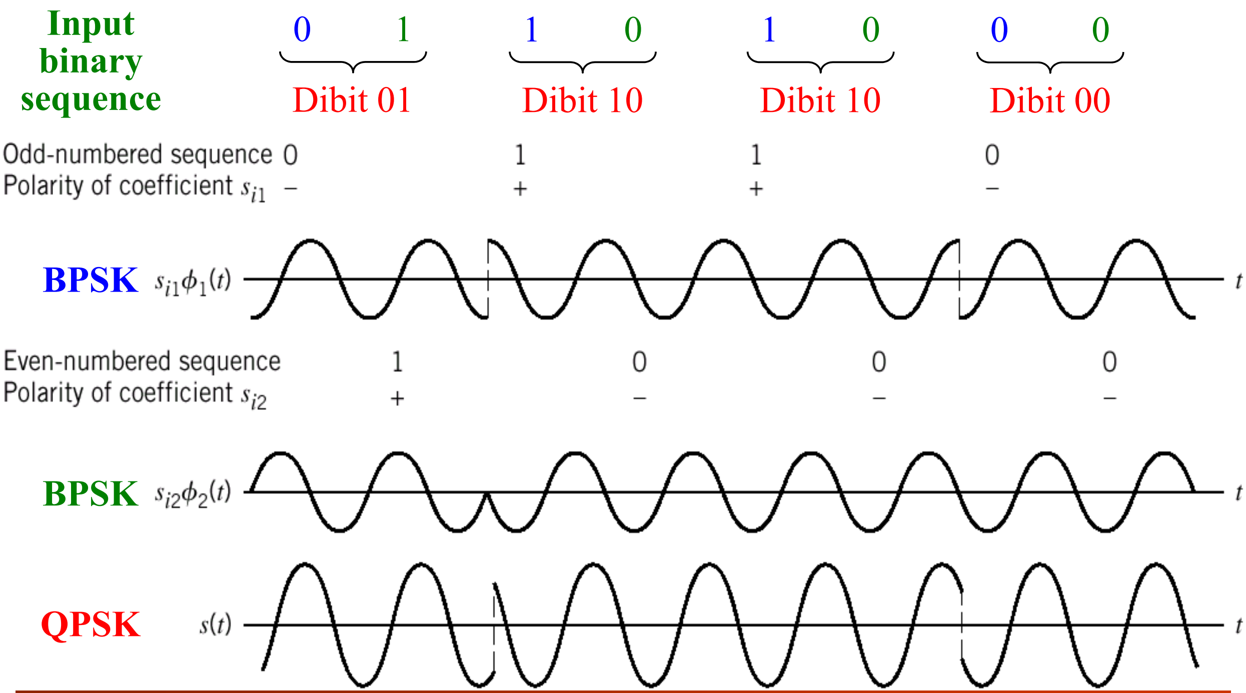
- According to the two **basis functions**, the complex plane is divided into four decision regions

$$\phi_1(t) = \sqrt{2/T}\cos(2\pi f_c t)$$
$$\phi_2(t) = \sqrt{2/T}\sin(2\pi f_c t)$$



Example 1: QPSK

- A QPSK signal can be regarded as two orthogonal BPSK signals



Error Probability of QPSK

- The observation vector **x** of QPSK has two elements

$$x_1 = \int_0^T x(t)\phi_1(t)\,dt = \sqrt{E}\cos\left[(2i-1)\pi/4\right] + w_1 = \pm\sqrt{E/2} + w_1$$

$$x_2 = \int_0^T x(t)\phi_2(t)\,dt = -\sqrt{E}\sin\left[(2i-1)\pi/4\right] + w_2 = \mp\sqrt{E/2} + w_2$$

- Based on the **ML decision rule**, the decision regions are

$$Z_1 : \begin{cases} 0 < x_1 < \infty \\ -\infty < x_2 < 0 \end{cases};\ Z_2 : \begin{cases} -\infty < x_1 < 0 \\ -\infty < x_2 < 0 \end{cases};\ Z_3 : \begin{cases} -\infty < x_1 < 0 \\ 0 < x_2 < \infty \end{cases};\ Z_4 : \begin{cases} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{cases}$$

- A coherent QPSK system is equivalent to **two** coherent BPSK systems ⇒ x_1 and x_2 can be viewed as **independent outputs**
- The equivalent coherent BPSK systems are characterized as
  - The **signal energy per bit** is $E/2$
  - The **noise variance** is $N_0/2$

Error Probability of QPSK (Cont.)

- According to the average bit error probability of **coherent BPSK**, the average probability of bit error in **each channel** is

$$P' = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E/2/N_0}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E/2N_0}\right)$$

- The average probability of a **correct decision** of **a symbol** is

$$P_c = (1-P')^2 = 1 - \operatorname{erfc}\left(\sqrt{E/2N_0}\right) + \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{E/2N_0}\right)$$

- The average probability of **symbol error** for coherent QPSK is

$$P_e = 1 - P_c = \operatorname{erfc}\left(\sqrt{E/2N_0}\right) - \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{E/2N_0}\right)$$

- In the region where $E/2N_0 >> 1$, we can ignore the **quadratic term**

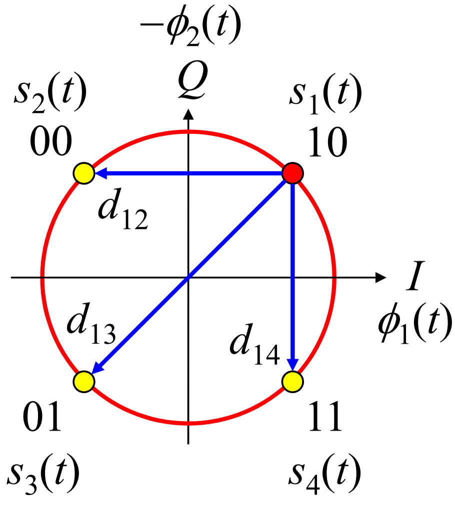
$$P_e \simeq \operatorname{erfc}\left(\sqrt{E/2N_0}\right)$$

Error Probability of QPSK (Union Bound)

- Another approach (**union bound**) of deriving the average probability of **symbol error** for coherent QPSK
  - The **sum** of all **pairwise** error probabilities
  - The error regions may be overlapped
  - It is an **upper bound**
- For the signal point $s_i(t)$

$$P_e \le \frac{1}{2}\sum_{k=1,k\neq i}^{4}\operatorname{erfc}\left(d_{ik}/2\sqrt{N_0}\right), \quad \text{for } \forall i$$

  - where d_{ik} is the distance between the two signal points $s_i(t)$ and $s_k(t)$



Bit Error Probability of QPSK

- If we consider only the set of **nearest** message points, a tighter approximated symbol error probability can be obtained
- Consider only the two **nearest** message points with the distances

$$d_{12} = d_{14} = \sqrt{2E}, \quad \text{for message point 1}$$

- The average probability of **symbol error** becomes

$$P_e \simeq 2 \times \frac{1}{2}\operatorname{erfc}\left(\sqrt{E/2N_0}\right) = \operatorname{erfc}\left(\sqrt{E/2N_0}\right)$$

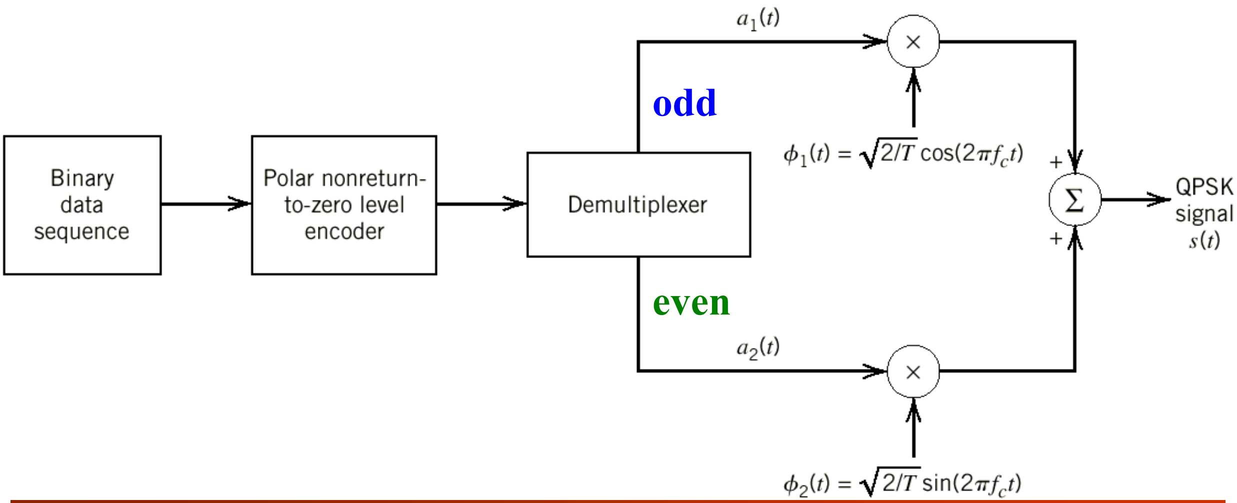
- Based on Gray encoding and the fact that $E = 2E_b$, the **bit error rate** of QPSK is

$$BER = \frac{1}{2}P_e \simeq \frac{1}{2}\operatorname{erfc}\left(\sqrt{E/2N_0}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/N_0}\right)$$

$$BER \text{ of coherent BPSK: } \; P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/N_0}\right)$$

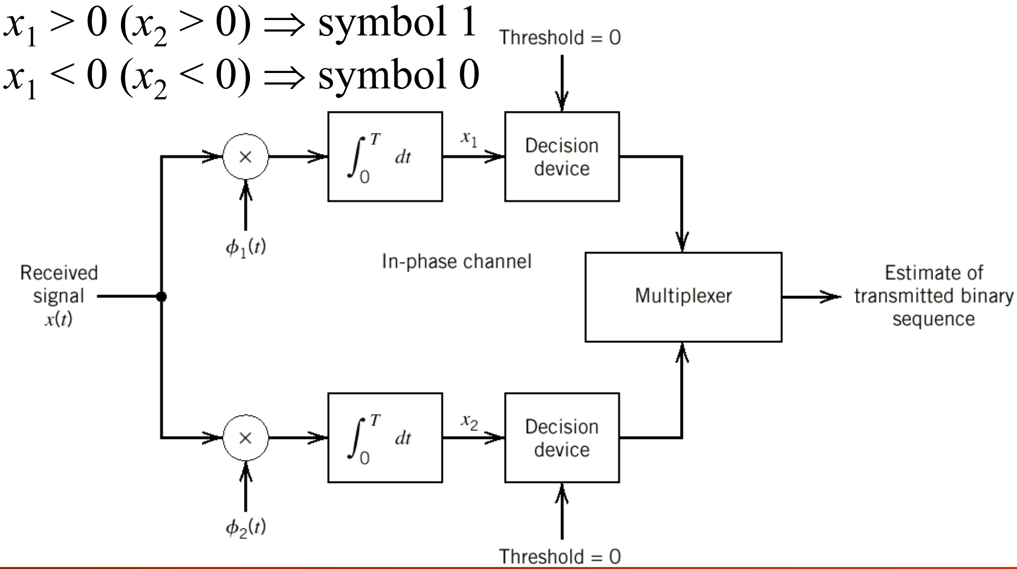
Generation of QPSK Signals

- The binary data sequence is transformed into the **polar form**
  - **NRZ** with symbols 1 and 0 represented by $+\sqrt{E_b}$ and $-\sqrt{E_b}$
- The binary wave is divided by a **demultiplexer** into two separate binary waves (**odd-** and **even-**numbered input bits)



Detection of QPSK Signals

- The received noisy QPSK signal $x(t)$ is applied to a pair of **correlators** (local **coherent** reference signal $\phi_1(t)$ and $\phi_2(t)$)
- The outputs x_1 and x_2 are compared with a zero-volt threshold
  - If $x_1 > 0$ $(x_2 > 0) \Rightarrow$ symbol 1
  - If $x_1 < 0$ $(x_2 < 0) \Rightarrow$ symbol 0



Power Spectra of QPSK Signals

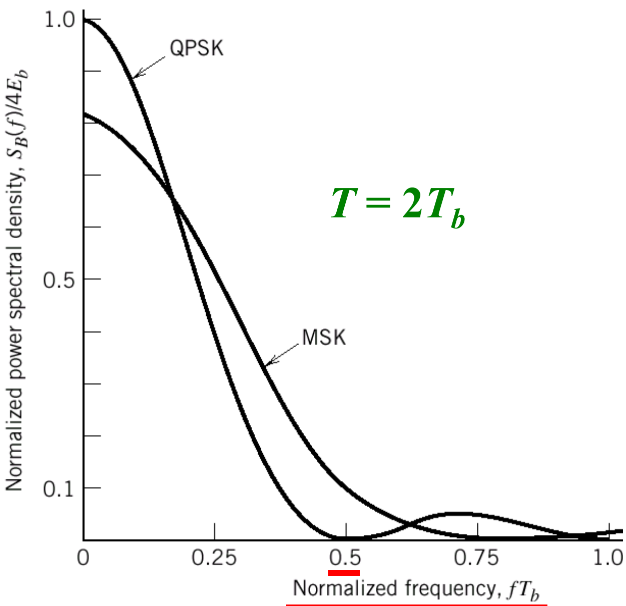
- The complex envelope of a QPSK signal consists of **in-phase** and **quadrature components**, both are equal to
  - Symbol 1: $+g(t)$
  - Symbol 0: $-g(t)$
- The symbol **shaping function**:

$$g(t) = \begin{cases} \sqrt{E/T}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

- The energy spectral density is

$$S_B(f) = 2E\,\text{sinc}^2(Tf) \\ = 4E_b\,\text{sinc}^2(2T_b f)$$

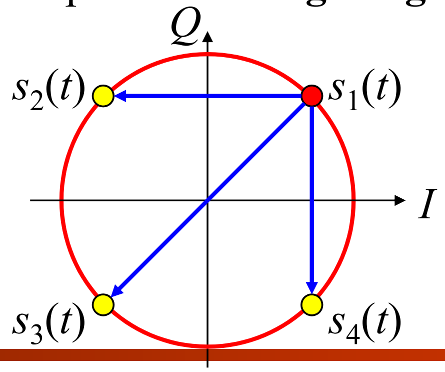
For BPSK
$S_B(f) = 2E_b\,\text{sinc}^2(T_b f)$



Offset QPSK

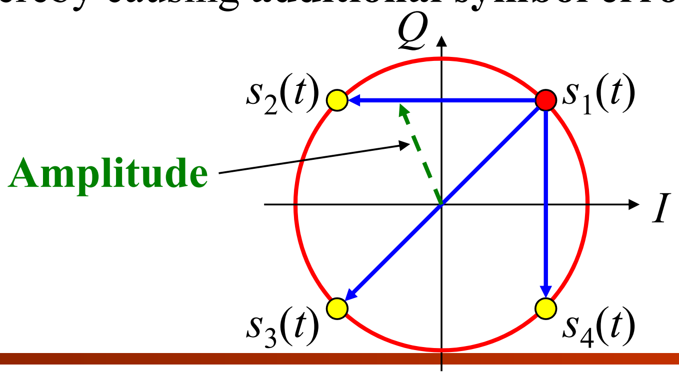
Phase Shift in QPSK Signals

- For ordinary QPSK, we have the following observations:
  - The carrier phase changes by **± 180°** whenever **both** the in-phase and quadrature components **change sign**
  - The carrier phase changes by **± 90°** whenever the in-phase **or** quadrature component **changes sign**
  - The carrier phase is **unchanged** when **neither** the in-phase **nor** quadrature components **changes sign**



Phase Shift in QPSK Signals (Cont.)

- The **180°** and **90°** phase shift result in **changes in the carrier amplitude** at the transmitter
  - The **carrier amplitude** is the distance between the **signal point** and the **origin**
  - Signal will be **distorted** because of **linearity limitation** of the power amplifier
  - Thereby causing **additional symbol errors** on detection



Offset QPSK (OQPSK)

- The OQPSK signals can change the carrier phase by only **± 90°**
  - The bit stream responsible for generating the **quadrature** component is **delayed by half a symbol interval**
- The phasor trajectory **does not** pass through **the origin**
- The two **basis functions** of offset QPSK are defined by

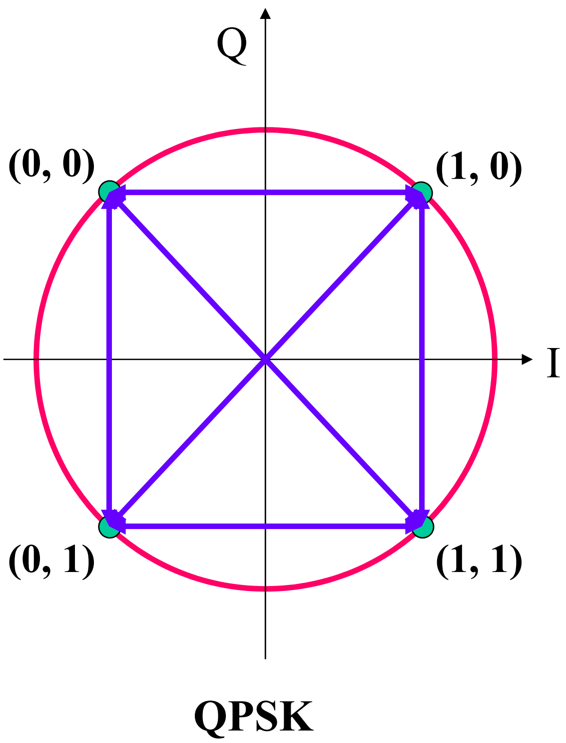
$$\phi_1(t) = \sqrt{2/T}\cos(2\pi f_c t), \quad 0 \le t < T$$
$$\phi_2(t) = \sqrt{2/T}\sin(2\pi f_c t), \quad \underline{T/2 \le t < 3T/2}$$

- The **bit error rate** in the in-phase or quadrature channel of a coherent QPSK system is still equal to

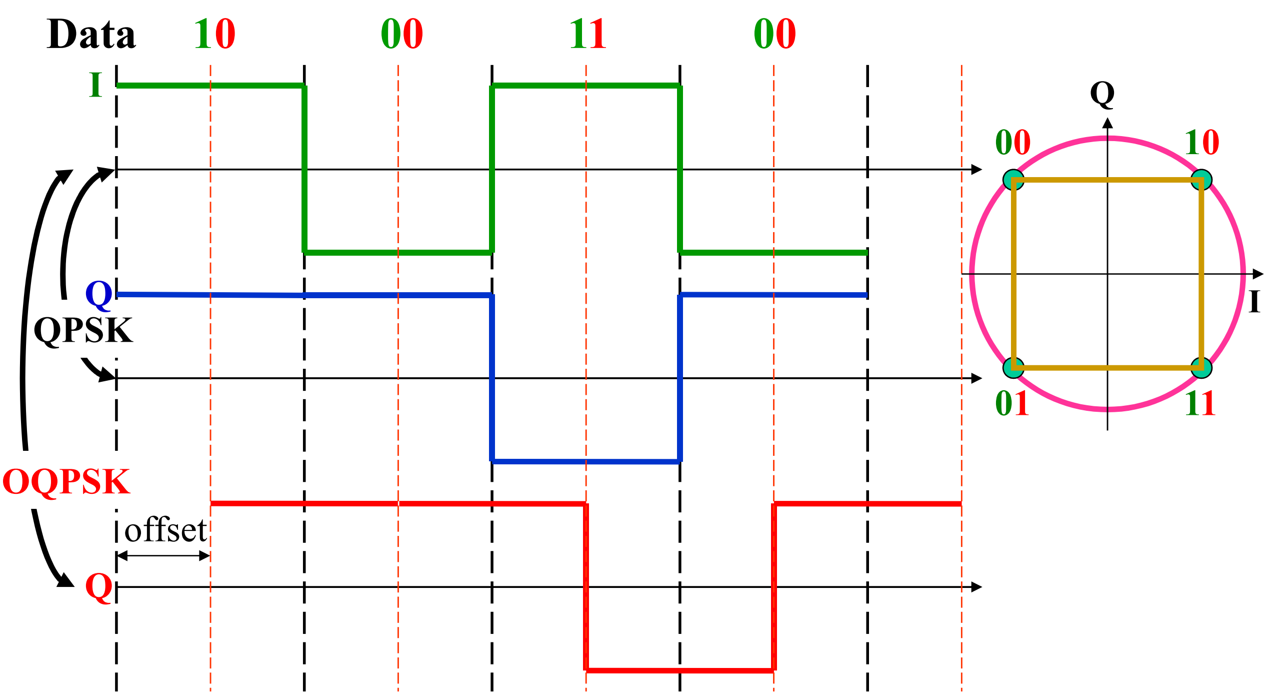
$$BER = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/N_0}\right)$$

  - The same as that of the conventional QPSK systems

Offset QPSK (OQPSK) (Cont.)



Offset QPSK (OQPSK) (Cont.)



M-ary PSK

M-ary PSK

- In M-ary PSK, the carrier takes on one of $\boldsymbol{M}$ **possible values**

$$\theta_i = 2(i-1)\pi/M\,, \quad i = 1, 2, \cdots, M$$

- The transmitted signal is defined as

$$s_i(t) = \sqrt{2E/T}\cos(2\pi f_c t + 2(i-1)\pi/M), \quad 0 \le t < T$$

- Each $s_i(t)$ can be expressed by the two basis functions

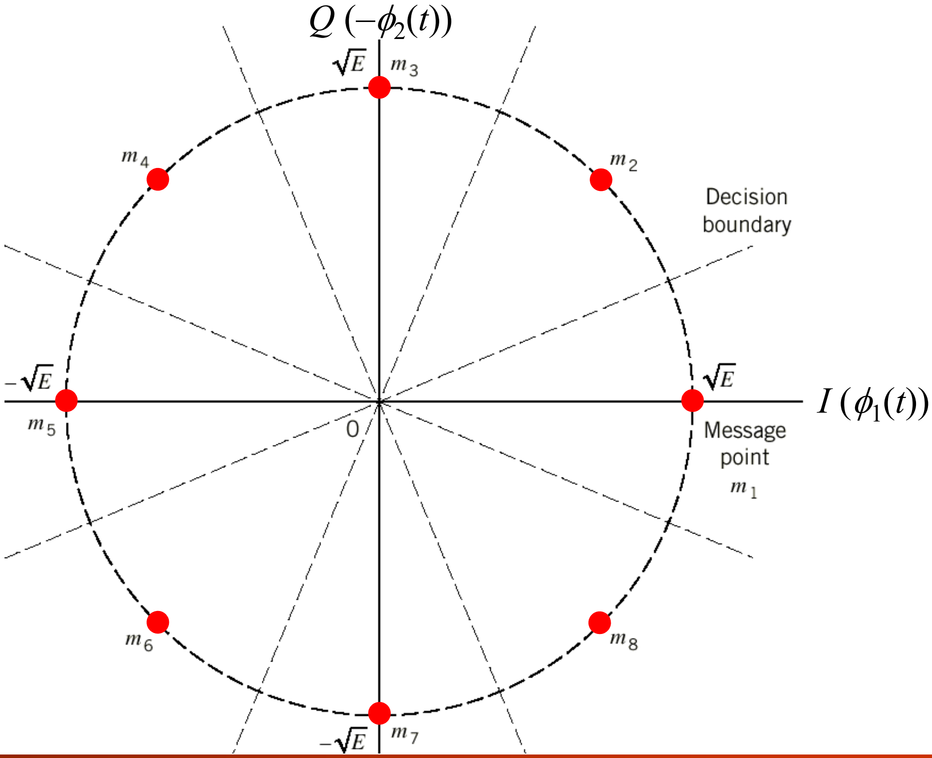
$$\phi_1(t) = \sqrt{2/T}\cos(2\pi f_c t); \quad \phi_2(t) = \sqrt{2/T}\sin(2\pi f_c t)$$

  - where the symbol duration $T = T_b \log_2 M$ ← **No. of bits in a symbol**
- The signal-space diagram is **circularly symmetric**
- The **average probability of symbol error** for M-ary PSK can be derived based on the **union bound**

$$P_e \le \frac{1}{2}\sum_{k=1,k\ne i}^{M} \text{erfc}\left(d_{ik}/2\sqrt{N_0}\right)$$

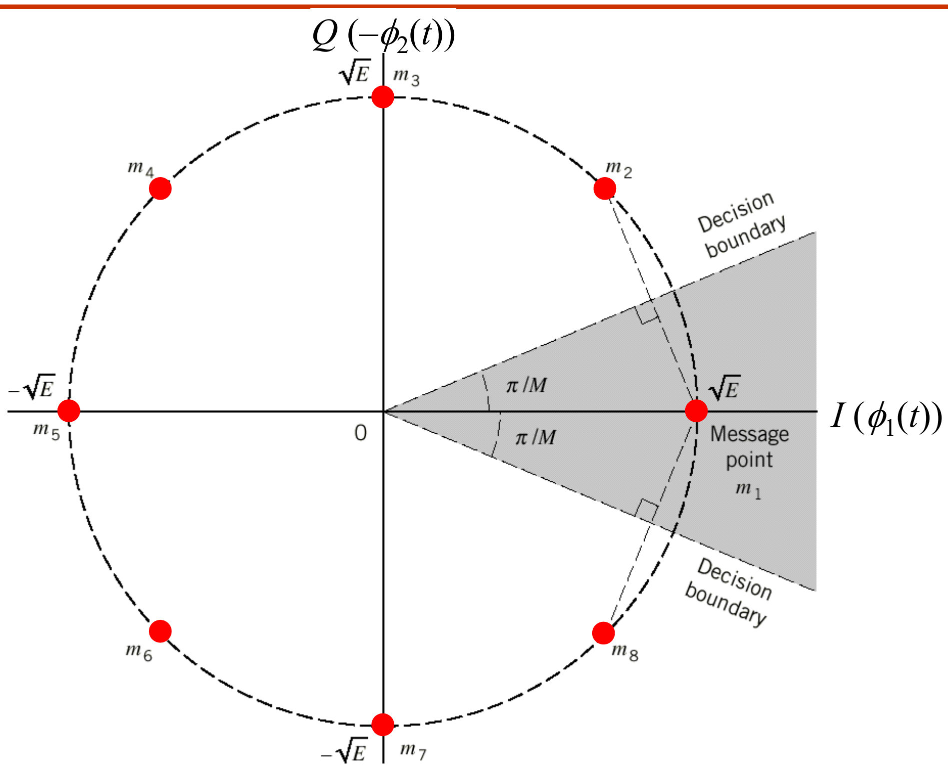
M-ary PSK (Cont.)

- $M = 8$



M-ary PSK (Cont.)

- $M = 8$



Error Probability of M-ary PSK

- Consider only the two **nearest** message points with the distances

$$d_{12} = d_{1M} = 2\sqrt{E}\sin(\pi/M), \quad \text{for message point 1}$$

- The average probability of **symbol error** becomes

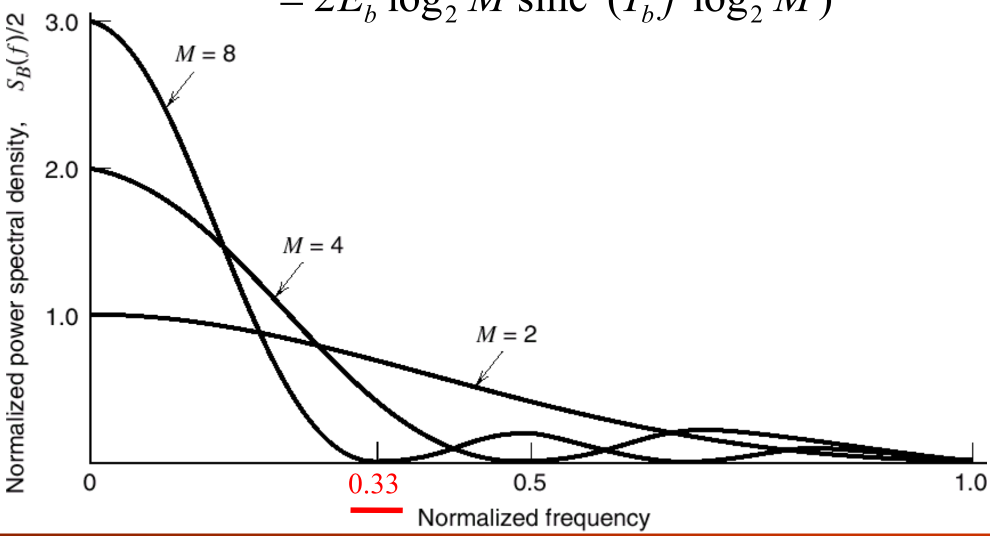
$$P_e \simeq \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/M)\right)$$

  – where it is assumed that $M \geq 4$

Power Spectra of M-ary PSK Signals

- The symbol duration of M-ary PSK is defined by $T = T_b \log_2 M$
- The energy spectral density is

$$S_B(f) = 2E\,\text{sinc}^2(Tf)$$
$$= 2E_b \log_2 M\,\text{sinc}^2(T_b f \log_2 M)$$



Bandwidth Efficiency of M-ary PSK Signals

- The **channel bandwidth** required to pass M-ary PSK signals (the **main spectral lobe** of M-ary PSK signals) is given by

$$B = 2/T$$

Bandwidth over the channel

- Since $R_b = 1/T_b$ and $T = T_b \log_2 M$, we may rewrite the channel bandwidth as

$$B = 2R_b/\log_2 M$$

- The bandwidth efficiency is given by

$$\rho = \frac{R_b}{B} = \frac{\log_2 M}{2}$$

- When M is **increased**, the bandwidth efficiency is **improved** at the cost of **degradation in the error performance**

M	2	4	8	16	32	64
ρ (bits/s/Hz)	0.5	1	1.5	2	2.5	3

Non-coherent Phase-Shift Keying

Non-coherent Phase-Shift Keying

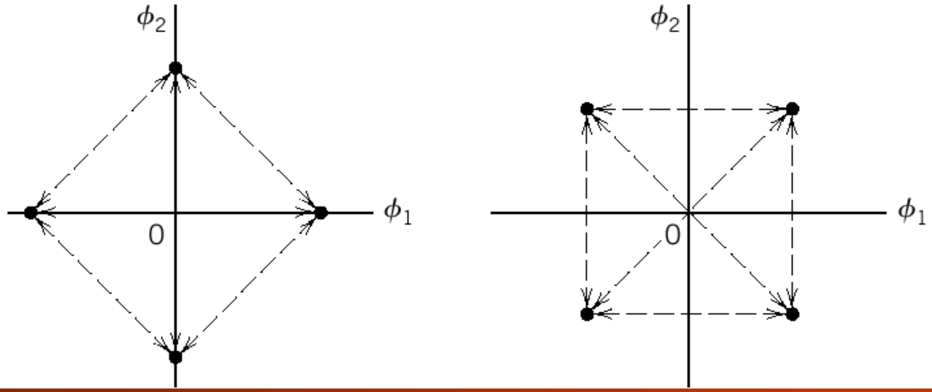
- In some communication channels, the **phase of the channel** cannot be determined easily
  - For example, the **wireless** communication channels
- How to obtain the phase information for **coherent detection**?
  - The transmitter may deliver a **reference signal** (with **fixed carrier phase known to the receiver**) for the receiver to acquire the phase information of the channel
- This approach does not always work well
  - It **degrades** the bandwidth efficiency
  - The phase of the channel may **change rapidly**
- If the phase information of the channel cannot be obtained
  - Use **non-coherent** modulation scheme

$$\theta \xrightarrow{\text{Channel}} \theta + \phi$$
$$\Rightarrow \text{Channel Phase: } \phi$$

π/4-Differential QPSK

π/4-Differential QPSK

- An ordinary QPSK signal may reside in **either one** of the two commonly used constellations
  - which are shifted by **π/4 radians** with respect to each other
- A π/4-Differential QPSK signal uses the two constellations **alternately** in **two successive symbols**
- The signal may reside in any one of **eight** possible phase states



π/4-Differential QPSK (Cont.)

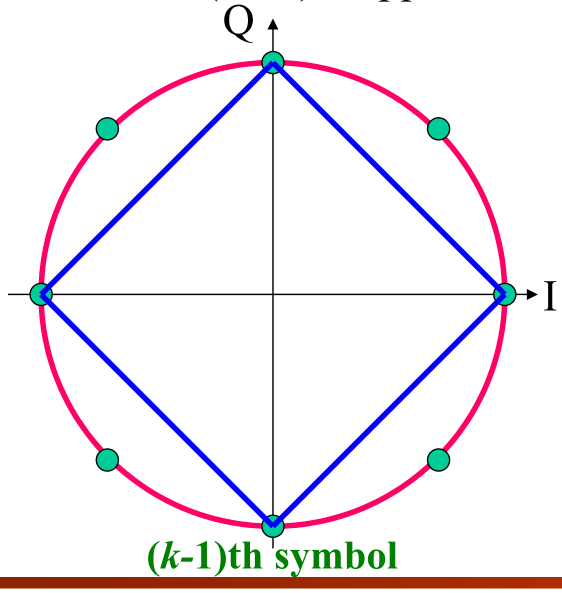
- Attractive features of π/4-Differential QPSK includes:
  - The **phase transitions** between the signals of two successive symbols are restricted to ± **π/4** and ± **3π/4** radians ⇒ **Less sensitive** to the **nonlinearity** of the power amplifier
  - π/4-Differential QPSK can be **noncoherently detected**
- The **generation** of π/4-Differential QPSK symbols follows the pair of relationships:

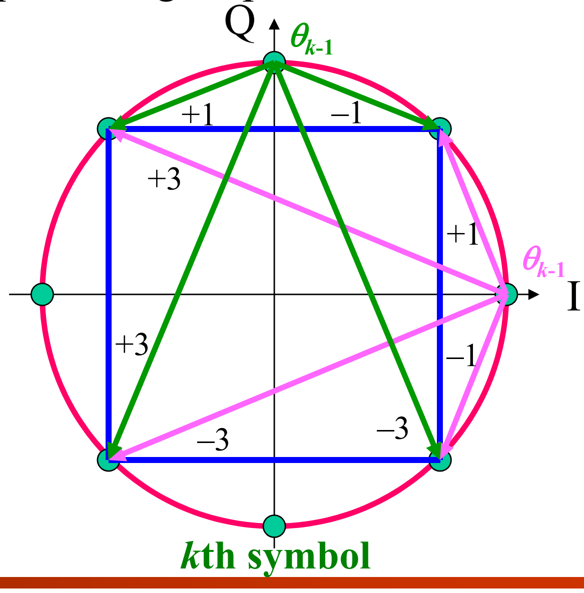
$$I_k = \cos(\theta_{k-1} + \Delta\theta_k) = \cos\theta_k; \quad Q_k = \sin(\theta_{k-1} + \Delta\theta_k) = \sin\theta_k$$

Gray-encoded Input Dibit	Phase Change, $\Delta\theta$ (radians)
00	π/4
01	3π/4
11	– 3π/4
10	– π/4

π/4-Differential QPSK (Cont.)

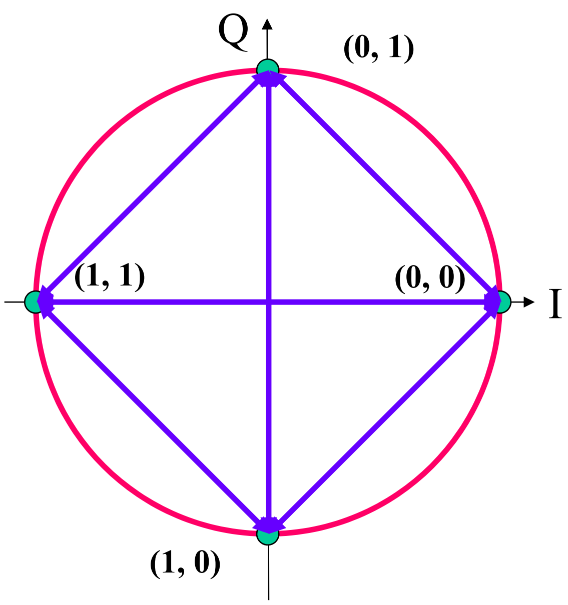
- There will certainly have a **phase transition** between the signals of **two successive symbols**
- The data (dibit) mapped to a specific signal point is **not fixed**



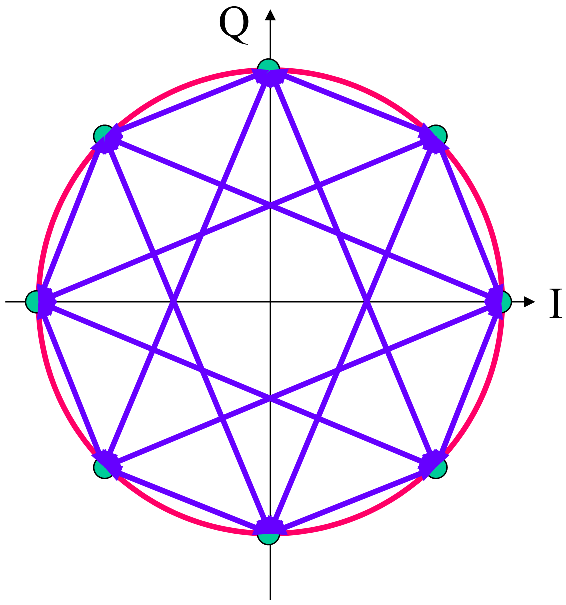


π/4-Differential QPSK (Cont.)

- The phasor trajectory does not pass through **the origin**



QPSK



π/4-DQPSK

Example 2

- The input binary sequence is **0 0 1 0 1 0 0 1**
- Suppose that the initial carrier phase is $\theta_0 = \pi/4$

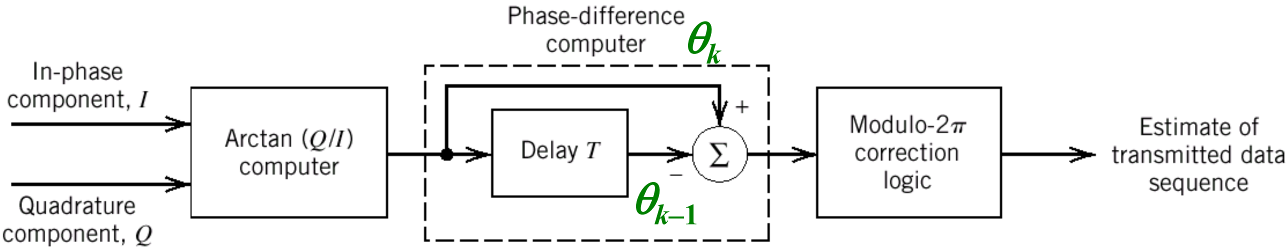
Step k	Phase θ_{k-1}	Input Dibit	Phase Change $\Delta\theta_k$	Transmitted Phase θ_k
1	$\pi/4$	00	$\pi/4$	$\boldsymbol{\pi/2}$
2	$\boldsymbol{\pi/2}$	10	$-\pi/4$	$\boldsymbol{\pi/4}$
3	$\boldsymbol{\pi/4}$	10	$-\pi/4$	**0**
4	**0**	01	$3\pi/4$	$3\pi/4$

Detection of π/4-DQPSK Signals

- The data information is not relied on the absolute signal phase
  - It relies on the relative **phase change** between two successive received symbols
  - **No** carrier phase information is required for data detection
- Another advantage of π/4-DQPSK modulation is that **symbol interval synchronization** is **easier** than conventional QPSK
  - There will certainly have a **phase transition** between the signals of **two successive symbols**
- The receiver first computes the **projections** of a noisy π/4-DQPSK signal $x(t)$ **onto the basis functions** $\phi_1(t)$ and $\phi_2(t)$
  - To extract the received signal phase

Detection of π/4-DQPSK Signals (Cont.)

- The resulting outputs, denoted by I and Q, are applied to a **differential detector** that consists of
  - **Arctangent computer:** extracting the phase of angle θ
  - **Phase-difference computer:** determining the change in the phase θ occurring **over one symbol interval**
  - **Modulo-2π correction logic:** correcting errors due to the possibility of **phase angles wrapping** around the real axis
    - To restrict the phase difference within (– π, + π)



Detection of π/4-DQPSK Signals (Cont.)

- Let $\Delta\theta_k$ denote the computed phase difference between θ_k and θ_{k-1} for the channel outputs of symbol k and $k – 1$
- The modulo-2π correction logic operates as follows:

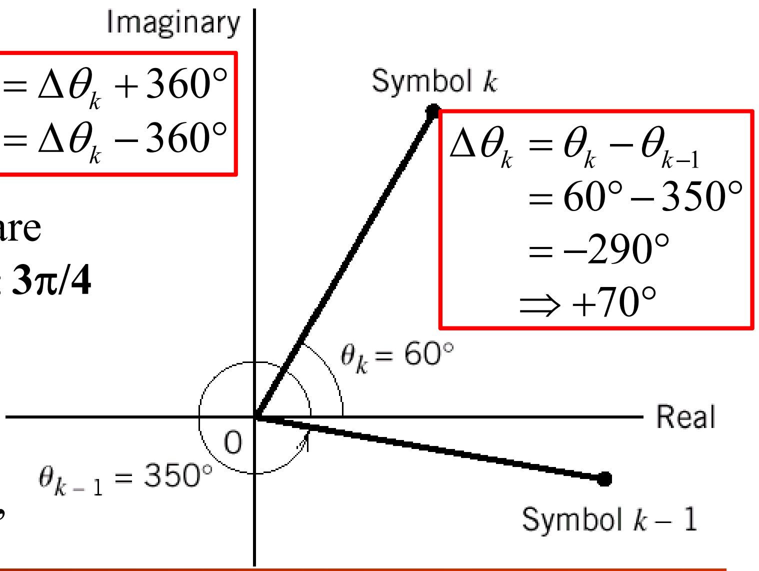
$$\text{If } \Delta\theta_k < -180^\circ \quad \text{Then } \Delta\theta_k = \Delta\theta_k + 360^\circ$$
$$\text{If } \Delta\theta_k > 180^\circ \quad \text{Then } \Delta\theta_k = \Delta\theta_k - 360^\circ$$

- The **phase transitions** are restricted to **± π/4** and **± 3π/4**

⇒ **$-180^\circ \le \Delta\theta_k \le +180^\circ$**

- If $\Delta\theta_k$ = + 70°, the detection result is + π/4

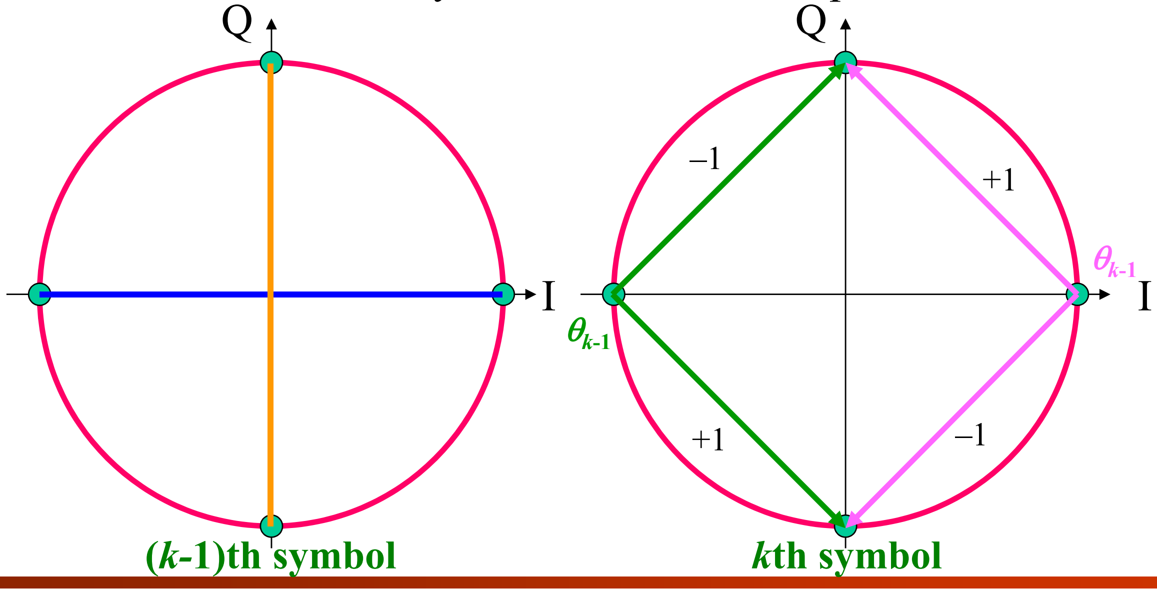
⇒ The decoded dibit: “00”



π/2-Differential BPSK

π/2-Differential BPSK

- A π/2-Differential BPSK signal uses two constellations **alternately** in **two successive symbols**
  - which are shifted by **π/2 radians** with respect to each other



π/2-Differential BPSK

- π/2-DBPSK with appropriate filtering can be used to approximate a precoded **Gaussian Minimum-Shift Keying** (GMSK)
  – which is a constant-envelope modulation
- GMSK will be introduced in other chapter

Homework

- **You must give detailed derivations or explanations, otherwise you get no points.**
- Communication Systems, Simon Haykin **(4^{th} Ed.)**
- 6.2;
- 6.5;
- 6.6;
- 6.10;